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The Jordanian Bicovariant Differential Calculus*

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Abstract

We show that the Woronowicz prescription using a bimodule constructed out of a tensorial product of a bimodule and its conjugate and a bi-covariant singlet leads to a trivial differential calculus.

1 Introduction

It is by now well known that our naive conception of the space-time as a collection of points equipped with suitable topological and metric structures at the energies much below the Planck scale should be modified. One possible approach to the description of physical phenomena at small distance is based on non-commutative geometry of the space-time [1]. In the quantum groups picture [2] the symmetry is described by noncommutative non-cocommutative \ast Hopf algebra. The connection with noncommutative differential geometry has been made by Woronowicz [3] who introduced the theory of bicovariant differential calculus. This turns out to be the appropriate way to describe quantum gauge theories. In this letter we show that this method leads to a trivial calculus in the case of the Jordanian group $U_h(2)$.

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2 The Jordanian Quantum Group $U_h(2)$

We recall that there are only two quantum group structures which admit a central determinant on space of 2×2 matrices: $GL_q(2)$ [4] and $GL_h(2)$ [5] (the deformation of $M(2)$ was considered and named ‘‘Jordanian’’ by Manin [6]). The continuous parameter h was introduced by Zakrzewski [7].

Let \mathcal{A} be the associative unital C -algebra generated by the linear transformations M^n_m ($n, m = 1, 2$)

$$M^n_m = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad (1)$$

the elements a, b, c, d satisfying the relations

$$\begin{aligned} [a, c] &= hc^2, & [b, a] &= h(a^2 - D_h), \\ [d, c] &= hc^2, & [d, b] &= h(D_h - d^2), \\ [a, d] &= h(d - a)c, & [b, c] &= h(ac + cd), \end{aligned} \quad (2)$$

where $D_h = ad - cb - hcd = ad - bc + hac$ is the Jordanian central determinant. The classical case is obtained by setting h equal to zero. The relations (2) are obtained by applying either the method of Faddeev et al. [8] namely by solving the monodromy equation $RM_1M_2 = M_2M_1R$ where $M_1 = M \otimes I$, $M_2 = I \otimes M$ and R is given in Eq. (6) or the method of Manin [6] using M as transformation matrix of the appropriate quantum planes.

The $U_h(2)$ is obtained by requiring that the unitary condition hold for this 2×2 matrix:

$$M_m^n{}^\dagger = M_m^n{}^{-1}. \quad (3)$$

The 2×2 matrix belonging to $U_h(2)$ preserves the nondegenerate bilinear form B_{nm} [9]

$$B_{nm}M_k^nM_l^m = D_hB_{kl}, \quad B^{nm}M_n^kM_m^l = D_hB^{kl}, \quad B_{kn}B^{nl} = \delta_k^l, \quad (4)$$

$$B_{nm} = \begin{pmatrix} 0 & -1 \\ 1 & h \end{pmatrix}, \quad B^{nm} = \begin{pmatrix} h & 1 \\ -1 & 0 \end{pmatrix}, \quad B^{nm}B_{nm} = -2. \quad (5)$$

3 $U_h(2)$ Woronowicz Bicovariant Differential Calculus

Zakrzewski [7] has applied the general construction of the Leningrad School [10] to the following R matrix which controls the noncommutativity of the elements M_m^n

$$R = \begin{pmatrix} 1 & -h & h & h^2 \\ 0 & 0 & 1 & -h \\ 0 & 1 & 0 & h \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (6)$$

The R matrix becomes the permutation operator $R^{nm}_{kl} = \delta_l^n \delta_k^m$ in the classical limit $h = 0$.

The braiding R matrix satisfy the Yang-Baxter equation

$$R^{ij}_{pq} R^{pk}_{lr} R^{qr}_{mn} = R^{jk}_{pq} R^{ip}_{rm} R^{rq}_{lm}. \quad (7)$$

The noncommutativity of the elements M_m^n is expressed as

$$R^{pq}_{nm} M_k^n M_l^m = M_n^p M_m^q R^{nm}_{kl}. \quad (8)$$

The algebra $Fun(U_h(2))$ is a Hopf algebra with comultiplication Δ , counit ϵ and antipode S which are given by:

- comultiplication (also called coproduct)

$$\Delta(M_m^n) = M_k^n \otimes M_m^k. \quad (9)$$

This coproduct Δ on $Fun(U_h(2))$ is directly related, for $h = 0$ (the non deformed case), to the pullback induced by left multiplication of the group on itself.

- co-unit ϵ

$$\epsilon(M_m^n) = \delta_m^n, \quad (10)$$

- antipode S (coinverse)

$$S(M_k^n) M_m^k = M_k^n S(M_m^k) = \delta_m^n, \quad (11)$$

$$S(M_m^n) = \frac{1}{D_h} B^{nk} M_k^l B_{lm}. \quad (12)$$

With the nondegenerate bilinear form B , the R matrix has the form

$$\begin{aligned} R^{+nm}_{kl} &= R^{nm}_{kl} = \delta^n_k \delta^m_l + B^{nm} B_{kl}, \\ R^{-nm}_{kl} &= R^{-1nm}_{kl} = R^{nm}_{kl}. \end{aligned} \quad (13)$$

The R matrix satisfies the Hecke relations $R^{\pm 2} = 1$ and the relation

$$B_{nm} R^{an}_{kc} R^{cm}_{lb} = \delta_b^a B_{kl}. \quad (14)$$

Now, we are going to consider the bicovariant bimodule Γ over $U_h(2)$. Let θ^a be a right invariant basis of Γ_{inv} , the linear subspace of all right -invariant elements of Γ i.e. $\Delta_R(\theta^a) = \theta^a \otimes I$. In the $h = 0$ the right coaction Δ_R coincides with the pullback for 1-forms. The left action is defined as

$$\Delta_L(\theta^a) = M^a_b \otimes \theta^b. \quad (15)$$

In the Jordanian quantum case we have $\theta^a M_m^n \neq M_m^n \theta^a$ in general, the bimodule structure of Γ being non-trivial for $h \neq 0$. There exist linear functionals $f_b^a : Fun(U_h(2)) \rightarrow \mathcal{C}$ for these left invariant basis such that

$$\theta^a M_m^n = (M_m^n * f_b^a) \theta^b = (f_b^a \otimes id) \Delta(M_m^n) \theta^b = f_b^a (M_m^n) M_m^k \theta^b. \quad (16)$$

Once we have the functionals f_b^a , we know how to commute elements of \mathcal{A} through elements of Γ . These functionals satisfy the consistent conditions:

$$\begin{aligned} f_b^a(M_m^n M_l^k) &= f_c^a(M_m^n) f_b^c(M_l^k) \\ f_b^a(I) &= \delta_b^a \\ (f_c^a \circ S) f_b^c &= \delta_b^a \epsilon; \quad f_c^a(f_b^c \circ S) = \delta_b^a \epsilon. \end{aligned} \quad (17)$$

Using these conditions, we find from Eq. (4) and Eq. (14) $f^a{}_b(M^n{}_k) = (D_h)^{\frac{1}{2}} R^{an}{}_{kb}$.

We can also define the conjugate basis $\theta^{*a} = (\theta^a)^* \equiv \bar{\theta}_a$.

The left coaction acts on these basis as

$$\Delta_L(\bar{\theta}_a) = S(M^b{}_a) \otimes \bar{\theta}_b. \quad (18)$$

This equation is easily obtained from Eq. (15) by the antilinear $*$ involution using the relations $(\Delta_R(\theta^a))^* = \Delta_R(\theta^a)^*$, $(M^n{}_m)^* = M^{*n}{}_m \equiv M^{\dagger n}{}_m = S(M^n{}_m)$. Then the linear functionals $\bar{f}^a{}_b$ are given by

$$\bar{\theta}_b M^n{}_m = \left(M^n{}_m * \bar{f}^a{}_b \right) \bar{\theta}_a, \quad (19)$$

where the functionals for the conjugated basis $\bar{\theta}_a$ is given by:

$$\bar{f}^a{}_b(S(M^n{}_m)) = (D_h)^{-\frac{1}{2}} R^{-an}{}_{mb}. \quad (20)$$

The representation with the upper index of $\bar{\theta}^a$ is defined by using the non-degenerate bilinear form B:

$$\bar{\theta}^b = \bar{\theta}_a B^{ab}. \quad (21)$$

This gives

$$\Delta_L(\bar{\theta}^a) = M^a{}_b \otimes \bar{\theta}^b, \quad (22)$$

which defines the new functionals $\tilde{f}^a{}_b$ corresponding to the basis $\bar{\theta}^a$

$$\tilde{f}^a{}_b = B_{bc} \bar{f}^c{}_d B^{da}. \quad (23)$$

We can easily find the transformation of the adjoint representation for the Jordanian quantum group which acts on the generators $M^n{}_m$ as the left coaction Ad_L :

$$Ad_L(M^n{}_m) = M^n{}_l S(M^k{}_m) \otimes M^l{}_k. \quad (24)$$

As usual, in order to define the bicovariant differential calculus with the $*$ -structure we have to require that the $*$ -operation is a bimodule antiautomorphism $(\Gamma_{Ad})^* = \Gamma_{Ad}$. We find the right invariant bases containing the adjoint representation. They are obtained by taking the tensor product $\theta^a \bar{\theta}_b \equiv \theta^a_b$ of two fundamental modules. The bimodule generated by these bases is closed under the $*$ -operation. Using the fact that $\Delta_L(\theta^a \bar{\theta}_b) = \Delta_L(\theta^a) \Delta_L(\bar{\theta}_b)$ we find the left coaction on the basis θ^a_b

$$\Delta_L(\theta^a_b) = S(M^a_c) M^d_b \otimes \theta^c_d. \quad (25)$$

In this basis the left coaction is given by

$$\Delta_L(\theta^{ab}) = \frac{1}{D_h} M^a_c M^b_d \otimes \theta^{cd}. \quad (26)$$

We can deduce the relation between the left and the right multiplication for this basis

$$\theta^{ab} M^n_m = (M^n_m * f_{Ad}{}^{ab}{}_{cd}) \theta^{cd} = f_{Ad}{}^{ab}{}_{cd} (M^n_k) M^k_m \theta^{cd}, \quad (27)$$

where

$$f_{Ad}{}^{ab}{}_{cd} = \tilde{f}^b_d * f^a_c. \quad (28)$$

The exterior derivative d is defined as

$$\begin{aligned} dM^n_m &= \frac{1}{\mathcal{N}} [X, M^n_m]_- = \theta^{ab} (M^n_m * \chi_{ab}) \\ &= \chi_{ab} (M^n_k) \theta^{ab} M^k_m, \end{aligned} \quad (29)$$

where $X = B_{ab} \theta^{ab} = -\theta^{12} + \theta^{21} + h\theta^{22}$ is the singlet representation of θ^{ab} and is both left and right co-invariant, $\mathcal{N} \in \mathcal{C}$ is the normalization constant which we take purely imaginary $N^* = -N$ and χ_{ab} are the quantum analogue of right-invariant vector fields.

Using (20), (23), (28)

$$\begin{aligned}
dM_m^n &= \frac{1}{\mathcal{N}} (B_{ab} \delta_m^k M_k^n \theta^{ab} - B_{cd} f_{Ad}{}^{cd}{}_{ab} (S(M_m^k)) M_k^n \theta^{ab}) \\
&= \frac{1}{\mathcal{N}} (B_{ab} \delta_m^k - B_{cd} R^{ct}{}_{ma} R^{dk}{}_{tb}) M_k^n \theta^{ab} = 0.
\end{aligned} \tag{30}$$

From (29) and (30) we deduce

$$\chi_{ab}(M_m^k) = 0. \tag{31}$$

We see that it is a trivial calculus $dM_m^n = 0$. To obtain a nontrivial calculi we have followed, in a recent paper [11], the Karimipour [12] method for our $4D$ calculus and constructed a Jordanian trace. This trace has permitted us to define an invariant $U_h(2)$ Yang-Mills Lagrangian. The Jordanian BRST and anti-BRST transformations [13] can also be carried out and will be reported in a future work.

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